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REVERSION OF POWER SERIES AND THE EXTENDED RANEY COEFFICIENTS

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Abstract. In direct as well as diagonal reversion of a system of power series, the reversion coefficients may be expressed as polynomials in the coefficients of the original power series. These polynomials have coefficients which are natural numbers (*Raney coefficients*). We provide a combinatorial interpretation for Raney coefficients. Specifically, each such coefficient counts a certain collection of ordered colored trees. We also provide a simple determinantal formula for Raney coefficients which involves multinomial coefficients.

Let $F_1, \ldots, F_n$ be polynomials in variables $x_1, \ldots, x_n$ with complex coefficients, where $n \geqslant 2$. Suppose, for each i, $F_i = x_i +$ higher degree terms and the Jacobian determinant of $F_1, \ldots, F_n$ is equal to 1. Then the *Jacobian Conjecture* [1], [9] asserts, in this case, that $x_1, \ldots, x_n$ are also polynomials in $F_1, \ldots, F_n$ with complex coefficients. This long-standing conjecture has not been solved even for $n = 2$. Since it can be proved that $x_1, \ldots, x_n$ are (formal) power series in $F_1, \ldots, F_n$ with complex coefficients, the Jacobian Conjecture asserts that these power series are really polynomials. This provides the motivation for this paper.

Let $F_1, \ldots, F_n$ be power series in variables $x_1, \ldots, x_n$ of the form $F_i = x_i +$ higher degree terms with indeterminate coefficients for each i. It is known (e.g. [2, Chapter III, Section 4.4, Proposition 5, p. 219]) that $F = (F_1, \ldots, F_n)$ has a (unique) compositional inverse, i.e., there exists $G = (G_1, \ldots, G_n)$ where each G_i is a power series in variables $x_1, \ldots, x_n$ such that $F \circ G = 1$ and $G \circ F = 1$, or equivalently, $F_i(G_1, \ldots, G_n) = x_i$ and $G_i(F_1, \ldots, F_n) = x_i$ for all i. There are various methods in the literature to find the coefficients of G_i. In this paper we shall present two new ones. Since each coefficient of G_i is a polynomial in the indeterminate coefficients of $F_1, \ldots, F_n$, it is enough to find the coefficients of these polynomials. We will refer to these coefficients as the *(extended) Raney coefficients*. In the first method, generating functions in infinitely many variables are used to show that each Raney coefficient has a combinatorial interpretation as the number of colored trees in a certain collection (Theorems 2.4 and 2.5). In

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the second method, Jacobi's residue formula, in a more general setting (Theorem 1.4), is used to obtain a determinantal formula for each Raney coefficient (Corollary 2.7). Consequently this gives a formula for counting certain colored forests. These generalize Raney's results in [7] where $n = 1$.

1. Formal Laurent series and Jacobi's residue formula

Let R be a commutative ring with identity and let $\mathbb{Z}$ denote the ring of integers. A *Laurent monomial* in variables $x_1, \dots, x_n$ is a power product $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where each exponent j_i is in $\mathbb{Z}$. The *degree* of this monomial is the sum of its exponents. A *homogeneous Laurent polynomial* over R in variables $x_1, \dots, x_n$ is an R-linear combination of (finitely many) Laurent monomials of the same degree. A *formal Laurent series* in variables $x_1, \dots, x_n$ with coefficients in R is an expression of the form

$$F = \sum_{k=-\infty}^{\infty} f_k, \tag{1.1}$$

where each f_k is either 0 or a homogeneous Laurent polynomial of degree k in $x_1, \dots, x_n$ with coefficients in R, and there exists an integer p such that $f_k = 0$ for all $k < p$. In this case we shall call f_k the *homogeneous component* of F of degree k. The *order* of a nonzero formal Laurent series F is defined to be the smallest integer p such that $f_p \neq 0$. In this case, we shall call f_p the *initial part* of F. The set $R(((x_1, \dots, x_n)))$ [1] of all formal Laurent series forms a commutative ring with identity under the obvious addition and multiplication. Note that if F and G have orders p and q respectively, then their product has order at least $p + q$. If, in addition, R is a field, then FG has order precisely $p + q$.

Remark. Let $R[[x_1, \dots, x_n]]_{x_1 x_2 \cdots x_n}$ denote the localization of the power series ring $R[[x_1, \dots, x_n]]$ at the multiplicative set $\{1, (x_1 x_2 \cdots x_n), (x_1 x_2 \cdots x_n)^2, \dots\}$. Note that if $n \geqslant 2$, then

$$R[[x_1, \dots, x_n]]_{x_1 x_2 \dots x_n} \subsetneqq R(((x_1, \dots, x_n))),$$

as $\sum_{k=0}^{\infty} x_1^{2k+1} x_2^{-k}$ is not in the former but in the latter; although

$$R[[x_1]]_{x_1} = R(((x_1))).$$

Lemma 1.1. *Let $F \in R(((x_1, \dots, x_n)))$ and let f_p be the initial part of F. Consider the following conditions.*

1. *F has a multiplicative inverse in $R(((x_1, \dots, x_n)))$.*
2. *f_p has a multiplicative inverse in $R(((x_1, \dots, x_n)))$.*
3. *f_p consists of a single term with invertible coefficient.*

Then $3 \Rightarrow 2 \Rightarrow 1$.

Proof. $3 \Rightarrow 2$. If $f_p = a x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ with a invertible in R, then f_p has multiplicative inverse $a^{-1} x_1^{-j_1} x_2^{-j_2} \cdots x_n^{-j_n}$ in $R(((x_1, \dots, x_n)))$.

$2 \Rightarrow 1$. If f_p is invertible in $R(((x_1, \dots, x_n)))$, then

$$F = \sum_{k=p}^{\infty} f_k = f_p \left[1 + f_p^{-1} \sum_{k=p+1}^{\infty} f_k \right] = f_p [1 - H]$$

[1] Thanks are due to Boo Barkee for this eye-catching notation.

where $H = (-1)f_p^{-1}\sum_{k=p+1}^{\infty} f_k$ is a formal Laurent series. Since $\mathrm{order}(f_p^{-1}) = -p$, it follows that $\mathrm{order}(H) \geqslant 1$. Hence $\mathrm{order}(H^k) \geqslant k$ and therefore $1+H+H^2+\cdots \in R(((x_1,\ldots,x_n)))$. Thus F is invertible with inverse $f_p^{-1}(1+H+H^2+\cdots)$. □

Remark. In case R is a field the three conditions above are equivalent.

The *residue* of a formal Laurent series F in $R(((x_1,\ldots,x_n)))$ is the coefficient of $x_1^{-1}x_2^{-1}\cdots x_n^{-1}$ in F. Note that residue is additive in the sense that if an infinite sum $\sum_{i\in I} G_i$ makes sense, then

$$\mathrm{residue}\left(\sum_{i\in I} G_i\right) = \sum_{i\in I}\mathrm{residue}(G_i). \tag{1.2}$$

In what follows we let

$$\frac{\partial(H_1,\ldots,H_n)}{\partial(x_1,\ldots,x_n)}$$

be the Jacobian determinant of $H_1,\ldots,H_n \in R(((x_1,\ldots,x_n)))$ with respect to $x_1,\ldots,x_n$.

Lemma 1.2 ([9, pp. 471–472]). *Let* $[\![H_1,\ldots,H_n]\!] = \dfrac{\partial(H_1,\ldots,H_n)}{\partial(x_1,\ldots,x_n)}$. *Then*

1. $[\![\ ,\ldots,\ ]\!]$ *is R-multilinear.*
2. $[\![\ ,\ldots,\ ]\!]$ *is alternating, i.e.,* $[\![H_1,\ldots,H_n]\!] = 0$ *if there exist* $i \neq j$ *such that* $H_i = H_j$.
3. $[\![\ ,\ldots,\ ]\!]$ *is anticommutative, i.e.,*
$$[\![H_1,\ldots,H_i,\ldots,H_j,\ldots,H_n]\!] = -[\![H_1,\ldots,H_j,\ldots,H_i,\ldots,H_n]\!].$$
4. *(Product Rule)* $[\![F_1G_1,H_2,\ldots,H_n]\!] = F_1[\![G_1,H_2,\ldots,H_n]\!] + G_1[\![F_1,H_2,\ldots,H_n]\!]$.
5. *(Power Rule) For any integer m,*
$$[\![H_1^m,H_2,\ldots,H_n]\!] = mH_1^{m-1}[\![H_1,H_2,\ldots,H_n]\!].$$
(Here we require that H_1 be invertible if m is negative.)
6. *If H is invertible, then* $[\![H^{-1},H,H_3,\ldots,H_n]\!] = 0$.

Lemma 1.3. $\mathrm{residue}\left(\dfrac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)}\right) = 0.$

Proof. By 1 of Lemma 1.2 and (1.2), it is enough to examine the case where each F_i consists of a single term:

$$F_i = a_i\, x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}.$$

Factoring $a_i x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}$ from the i^{th} row of the Jacobian matrix for all i, and then factoring x_j^{-1} from the j^{th} column for all j, we have

$$\frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} = a_1\cdots a_n \begin{vmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{vmatrix} x_1^{-1+\sum b_{i1}} \cdots x_n^{-1+\sum b_{in}}.$$

Suppose $\sum b_{i1} = \cdots = \sum b_{in} = 0$. Then the displayed determinant is 0 and therefore the residue is 0. Otherwise, at least one of the x_i's has exponent $\neq -1$ and so the residue is 0 by definition. □

Theorem 1.4 (Jacobi's Residue Formula). *Suppose, for each $i = 1, \dots, n$, $F_i \in R(((x_1, \dots, x_n)))$ is of the form*

$$F_i = a_i\, x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}} + \text{higher degree terms}$$

with a_i invertible in R. Then

$$\operatorname{residue}\left(F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}\right) = \begin{cases} \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{vmatrix}, & \text{if } e_1 = e_2 = \dots = e_n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first consider the case $e_1 = e_2 = \dots = e_n = -1$. Following the proof of Lemma 1.3, we have

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} = a_1 \cdots a_n \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{vmatrix} x_1^{-1+\sum b_{i1}} \cdots x_n^{-1+\sum b_{in}} + \text{ higher degree terms.}$$

Since a_i is invertible, by the proof of Lemma 1.1, we have

$$F_i^{-1} = a_i^{-1}\, x_1^{-b_{i1}}\, x_2^{-b_{i2}} \cdots x_n^{-b_{in}} + \text{ higher degree terms}$$

for each i. Hence the result follows.

Consider now the remaining case. Permuting the F_i and using 3 of Lemma 1.2, we may assume that $e_1 \neq -1, \dots, e_j \neq -1$, but $e_{j+1} = \dots = e_n = -1$, for some j, $1 \leqslant j \leqslant n$. Setting $G_i = \frac{1}{e_i+1} F_i^{e_i+1}$ and using 5, 1 and, successively, 4, 6 and 3 of Lemma 1.2, we have

$$\begin{aligned} &F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n} [\![F_1, \dots, F_n]\!] \\ &\quad = \frac{1}{e_1+1} \cdots \frac{1}{e_j+1} F_{j+1}^{-1} \cdots F_n^{-1} [\![F_1^{e_1+1}, \dots, F_j^{e_j+1}, F_{j+1}, \dots, F_n]\!] \\ &\quad = F_{j+1}^{-1} \cdots F_n^{-1} [\![G_1, \dots, G_j, F_{j+1}, \dots, F_n]\!] \\ &\quad = F_{j+2}^{-1} \cdots F_n^{-1} [\![F_{j+1}^{-1} G_1, \dots, G_j, F_{j+1}, \dots, F_n]\!] \\ &\quad = F_{j+3}^{-1} \cdots F_n^{-1} [\![F_{j+1}^{-1} F_{j+2}^{-1} G_1, \dots, G_j, F_{j+1}, \dots, F_n]\!] \\ &\quad\ \cdots\cdots\cdots\cdots \\ &\quad = [\![F_{j+1}^{-1} F_{j+2}^{-1} F_{j+3}^{-1} \cdots F_n^{-1} G_1, \dots, G_j, F_{j+1}, \dots, F_n]\!]. \end{aligned}$$

The result now follows from Lemma 1.3. ☐

Corollary 1.5. *Suppose, for each $i = 1, \dots, n$, $F_i \in R(((x_1, \dots, x_n)))$ is of the form*

$$F_i = a_i x_i + \text{ higher degree terms}$$

with a_i invertible in R. Suppose also that for nonnegative integers $p_1, \dots, p_n$,

$$x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} = \sum_{l_1, \dots, l_n} d_{l_1, \dots, l_n}^{p_1, \dots, p_n} F_1^{l_1} F_2^{l_2} \cdots F_n^{l_n}, \tag{1.3}$$

where $d^{p_1,\ldots,p_n}_{l_1,\ldots,l_n} \in R$. *Then*

$$d^{p_1,\ldots,p_n}_{m_1,\ldots,m_n} = \text{residue}\left(x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} F_1^{-m_1-1} \cdots F_n^{-m_n-1} \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} \right).$$

Proof.

$$\begin{aligned}
&\text{residue}\left(x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} F_1^{-m_1-1} \cdots F_n^{-m_n-1} \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} \right) \\
&= \text{residue}\left(\sum_{l_1,\ldots,l_n} d^{p_1,\ldots,p_n}_{l_1,\ldots,l_n} F_1^{l_1-m_1-1} \cdots F_n^{l_n-m_n-1} \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} \right) \quad \text{by (1.3)} \\
&= \text{residue}\left(d^{p_1,\ldots,p_n}_{m_1,\ldots,m_n} F_1^{-1} \cdots F_n^{-1} \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} \right) \quad \text{by (1.2) and Theorem 1.4} \\
&= d^{p_1,\ldots,p_n}_{m_1,\ldots,m_n} \quad \text{by Theorem 1.4.} \quad \square
\end{aligned}$$

2. Power series reversion in two variables

A *2-colored tree* is a (finite) rooted tree in which every node has either color 1 or color 2 and the children of each node are linearly ordered in such a way that nodes of color 1 always precede those of color 2. A (p,q)-*forest* F is an ordered set of p 2-colored trees with color-1 roots followed by q 2-colored trees with color-2 roots. For instance, a $(1,0)$-forest is simply a 2-colored tree whose root has color 1.

Let $\mathbb{N}$ denote the set of nonnegative integers and let $\boldsymbol{\alpha} = (\alpha_{ij})^{i=0,1,2,\ldots}_{j=0,1,2,\ldots}$, $\boldsymbol{\beta} = (\beta_{ij})^{i=0,1,2,\ldots}_{j=0,1,2,\ldots}$ be two matrices over $\mathbb{N}$ with only finitely many nonzero entries. We shall call the ordered pair $(\boldsymbol{\alpha},\boldsymbol{\beta})$ the *inventory* of F if α_{ij} equals the number of color-1 nodes in F with i children of color 1 and j children of color 2, and β_{ij} equals the number of color-2 nodes in F with i children of color 1 and j children of color 2.

Let

$$\sigma(\boldsymbol{\alpha}) = \sum_{i,j} \alpha_{ij}, \quad \sigma_1(\boldsymbol{\alpha}) = \sum_{i,j} i\alpha_{ij},$$

and

$$\sigma_2(\boldsymbol{\alpha}) = \sum_{i,j} j\alpha_{ij}.$$

Example 2.0. Consider the 2-colored tree T in Figure 2.1. A round node indicates that it is of color 1 (or female) while a square node indicates that it is of color 2 (or male). Then α_{ij} counts the number of female nodes with i daughters and j sons, and β_{ij} counts the number of male nodes with i daughters and j sons.

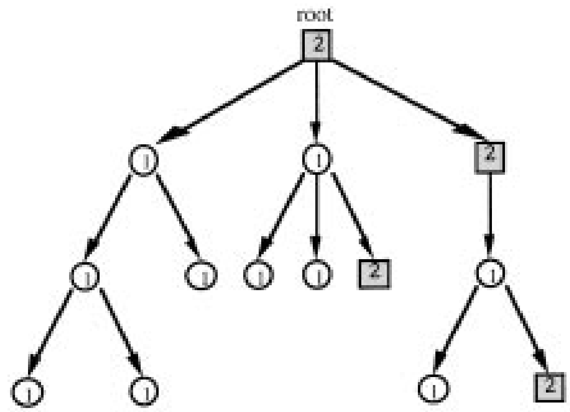

FIGURE 2.1

The inventory of the tree T is

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots & \vdots & \beta_{00} & \beta_{01} & \beta_{02} & \dots \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \vdots & \beta_{10} & \beta_{11} & \beta_{12} & \dots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \dots & \vdots & \beta_{20} & \beta_{21} & \beta_{22} & \dots \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & & \vdots & 2 & 0 & \\ 0 & 1 & \mathbf{0} & \vdots & 1 & 0 & \mathbf{0} \\ 2 & 1 & & \vdots & 0 & 1 & \\ \mathbf{0} & & \mathbf{0} & \vdots & \mathbf{0} & & \mathbf{0} \end{pmatrix}.$$

Lemma 2.1. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the inventory of a (p, q)-forest F, then*

$$p = \sigma(\boldsymbol{\alpha}) - \sigma_1(\boldsymbol{\alpha} + \boldsymbol{\beta}), \qquad q = \sigma(\boldsymbol{\beta}) - \sigma_2(\boldsymbol{\alpha} + \boldsymbol{\beta}). \tag{2.1}$$

Proof. Note that $\sum \alpha_{ij}$ is the number of color-1 nodes in F and $\sum \beta_{ij}$ is the number of color-2 nodes in F. Since the number of children of color 1 is equal to $\sum i\alpha_{ij} + \sum i\beta_{ij}$, the first identity follows from the fact that roots are the only nodes that are not children. The second identity can be proved similarly. □

Remark. The converse of the above is true if $pq \neq 0$ (see Corollary 3.4). To see that the condition $pq \neq 0$ is necessary, let $p = 1$, $q = 0$, $\alpha_{00} = 1$, $\beta_{01} = 1$ with all other α_{ij}'s and β_{ij}'s being zero. Then (2.1) is satisfied but $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is not the inventory of any $(1, 0)$-forest.

Given a pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of matrices as before, define $R(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to be the number of (p, q)-forests with inventory $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where p and q are defined by (2.1). For $p \geqslant 0$ and $q \geqslant 0$, we shall denote by $\mathcal{F}_{p,q}$ the set of all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfying (2.1). Let X and

Y be the generating functions for the number of $(1,0)$-forests and the number of $(0,1)$-forests respectively, both classified by inventories. Then

$$
\begin{aligned}
X &= \sum_{\mathcal{F}_{1,0}} R(\boldsymbol{\alpha},\boldsymbol{\beta}) \prod a_{ij}^{\alpha_{ij}} \prod b_{ij}^{\beta_{ij}}, \\
Y &= \sum_{\mathcal{F}_{0,1}} R(\boldsymbol{\alpha},\boldsymbol{\beta}) \prod a_{ij}^{\alpha_{ij}} \prod b_{ij}^{\beta_{ij}}.
\end{aligned} \tag{2.2}
$$

The sum in the first equation should be over all inventories of $(1,0)$-forests; however, it can be extended to $\mathcal{F}_{1,0}$ for if $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is in $\mathcal{F}_{1,0}$ but is not the inventory of any $(1,0)$-forest, then $R(\boldsymbol{\alpha},\boldsymbol{\beta})=0$. Both X and Y are formal power series in infinitely many variables a_{ij}, b_{ij}, where $i,j=0,1,2,\ldots$.

Lemma 2.2. *X^pY^q is the generating function for the number of (p,q)-forests classified by inventories. In fact,*

$$X^pY^q = \sum_{\mathcal{F}_{p,q}} R(\boldsymbol{\alpha},\boldsymbol{\beta}) \prod a_{ij}^{\alpha_{ij}} \prod b_{ij}^{\beta_{ij}}.$$

Proof. Let $(\boldsymbol{\alpha},\boldsymbol{\beta})$ be the inventory of a (p,q)-forest. Then $R(\boldsymbol{\alpha},\boldsymbol{\beta})$ is equal to $\sum R(\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1)\cdots R(\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_{p+q})$ where the sum is over all sequences $(\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1), \ldots,$ $(\boldsymbol{\alpha}_{p+q},\boldsymbol{\beta}_{p+q})$ such that $\boldsymbol{\alpha}=\sum\boldsymbol{\alpha}_k$, $\boldsymbol{\beta}=\sum\boldsymbol{\beta}_k$, and $(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)$ is the inventory of a $(1,0)$-forest for $k=1,\ldots,p$, $(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)$ is the inventory of a $(0,1)$-forest for $k=p+1,$ $\ldots, p+q$. Clearly, the sum can be extended to all sequences such that $\boldsymbol{\alpha}=\sum\boldsymbol{\alpha}_k$, $\boldsymbol{\beta}=\sum\boldsymbol{\beta}_k$, and $(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)$ is in $\mathcal{F}_{1,0}$ for $k=1,\ldots,p$, $(\boldsymbol{\alpha}_k,\boldsymbol{\beta}_k)$ is in $\mathcal{F}_{0,1}$ for $k=p+1,$ $\ldots, p+q$. Thus it follows from (2.2) that X^pY^q is the generating function for the number of (p,q)-forests. □

Lemma 2.3. *X and Y of (2.2) satisfy the following functional equations*

$$
\begin{aligned}
X &= \sum a_{ij}X^iY^j, \\
Y &= \sum b_{ij}X^iY^j.
\end{aligned} \tag{2.3}
$$

Proof. Using Lemma 2.2, we see that $a_{pq}X^pY^q$ is the generating function for the number of $(1,0)$-forests the removal of whose roots produces (p,q)-forests. Then the first equation follows by partitioning the set of all $(1,0)$-forests into classes by the number p of color-1-children and the number q of color-2-children of the roots. The proof for the second is similar. □

Theorem 2.4 (Direct Reversion). *Suppose*

$$
\begin{aligned}
F &= x - \sum_{i+j\geqslant 2} a_{ij}\,x^i\,y^j, \\
G &= y - \sum_{i+j\geqslant 2} b_{ij}\,x^i\,y^j,
\end{aligned} \tag{2.4}
$$

where the a_{ij} and the b_{ij} are indeterminates. Then, for $p,q\geqslant 0$,

$$x^py^q = \sum_{l,m} e_{l,m}^{p,q}\,F^lG^m$$

where

$$e_{l,m}^{p,q} = \sum R(\boldsymbol{\alpha},\boldsymbol{\beta}) \prod_{i+j\geqslant 2} a_{ij}^{\alpha_{ij}} \prod_{i+j\geqslant 2} b_{ij}^{\beta_{ij}}$$

with the sum indexed by all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *in* $\mathcal{F}_{p,q}$ *satisfying the condition that* $l = \alpha_{00}$, $m = \beta_{00}$, *and* $\alpha_{10} = \alpha_{01} = \beta_{10} = \beta_{01} = 0$

Proof. Setting $a_{01} = a_{10} = b_{01} = b_{10} = 0$, $X = x$, $Y = y$, $a_{00} = F$, $b_{00} = G$ in (2.2), and using Lemma 2.3, we see that the following satisfies (2.4).

$$
\begin{aligned}
x &= \sum_{l,m} \left(\sum R(\boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{i+j \geqslant 2} a_{ij}^{\alpha_{ij}} \prod_{i+j \geqslant 2} b_{ij}^{\beta_{ij}} \right) F^l \, G^m, \\
y &= \sum_{l,m} \left(\sum R(\boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{i+j \geqslant 2} a_{ij}^{\alpha_{ij}} \prod_{i+j \geqslant 2} b_{ij}^{\beta_{ij}} \right) F^l \, G^m,
\end{aligned} \tag{2.5}
$$

where the first inner sum is indexed by all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in $\mathcal{F}_{1,0}$ and the second by $\mathcal{F}_{0,1}$, and in both cases, satisfying the conditions: $l = \alpha_{00}$, $m = \beta_{00}$ and $\alpha_{10} = \alpha_{01} = \beta_{10} = \beta_{01} = 0$. By [2, Chapter III, Section 4.4, Proposition 5, pp. 219–220], a right compositional inverse is also a left inverse in formal power series reversion, so (2.4) also satisfies (2.5). Now the result follows from Lemma 2.2. □

Theorem 2.5 (Diagonal Reversion). *Suppose*

$$
F = \frac{x}{1 + \sum_{i+j \geqslant 1} a_{ij} \, x^i \, y^j}, \qquad G = \frac{y}{1 + \sum_{i+j \geqslant 1} b_{ij} \, x^i \, y^j},
$$

where the a_{ij} *and the* b_{ij} *are indeterminates. Then, for* $p, q \geqslant 0$,

$$
x^p y^q = \sum_{l,m} d_{l,m}^{p,q} \, F^l G^m
$$

where

$$
d_{l,m}^{p,q} = \sum R(\boldsymbol{\alpha}, \boldsymbol{\beta}) \prod_{i+j \geqslant 1} a_{ij}^{\alpha_{ij}} \prod_{i+j \geqslant 1} b_{ij}^{\beta_{ij}}
$$

with the sum indexed by all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *in* $\mathcal{F}_{p,q}$ *satisfying the conditions* $l = \sigma(\boldsymbol{\alpha})$ *and* $m = \sigma(\boldsymbol{\beta})$.

Proof. Substitute $a_{00}a_{ij}$ for a_{ij}, $b_{00}b_{ij}$ for b_{ij} for all $i + j \geqslant 1$, and then set $a_{00} = F$, $b_{00} = G$ in (2.2). Now proceed as in the proof of Theorem 2.4. □

In the following we shall find a formula for $R(\boldsymbol{\alpha}, \boldsymbol{\beta})$ thus determining $e_{l,m}^{p,q}$ and $d_{l,m}^{p,q}$ in Theorems 2.4 and 2.5 respectively. For any matrix $\boldsymbol{\alpha} = (\alpha_{ij})_{j=0,1,2,\ldots}^{i=0,1,2,\ldots}$ over $\mathbb{N}$ with a finite number of nonzero entries, we define the multinomial and the "modified" multinomial coefficients by

$$
\begin{aligned}
M(\boldsymbol{\alpha}) &= \binom{\sigma(\boldsymbol{\alpha})}{\alpha_{00}, \ldots, \alpha_{ij}, \ldots} = \frac{\sigma(\boldsymbol{\alpha})!}{\prod (\alpha_{ij}!)}, \\
\widetilde{M}(\boldsymbol{\alpha}) &= \begin{cases} 1, & \text{if all } \alpha_{ij} = 0, \\ \dfrac{1}{\sigma(\boldsymbol{\alpha})} M(\boldsymbol{\alpha}), & \text{otherwise.} \end{cases}
\end{aligned}
$$

Theorem 2.6. *Suppose F and G are as in Theorem 2.5. Then*

$$d_{l,m}^{p,q} = \sum \left| \begin{pmatrix} M(\boldsymbol{\alpha}) & 0 \\ 0 & M(\boldsymbol{\beta}) \end{pmatrix} - \begin{pmatrix} \widetilde{M}(\boldsymbol{\alpha}) & 0 \\ 0 & \widetilde{M}(\boldsymbol{\beta}) \end{pmatrix} \begin{pmatrix} \sigma_1(\boldsymbol{\alpha}) & \sigma_2(\boldsymbol{\alpha}) \\ \sigma_1(\boldsymbol{\beta}) & \sigma_2(\boldsymbol{\beta}) \end{pmatrix} \right| \times \prod_{i+j\geqslant 1} a_{ij}^{\alpha_{ij}} \prod_{i+j\geqslant 1} b_{ij}^{\beta_{ij}}$$

with the sum indexed by all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in $\mathcal{F}_{p,q}$ satisfying the conditions $l = \sigma(\boldsymbol{\alpha})$ and $m = \sigma(\boldsymbol{\beta})$.

Proof. It suffices to prove

$$\begin{aligned} x^p y^q & \begin{vmatrix} F^{-l-1}\dfrac{\partial F}{\partial x} & F^{-l-1}\dfrac{\partial F}{\partial y} \\ G^{-m-1}\dfrac{\partial G}{\partial x} & G^{-m-1}\dfrac{\partial G}{\partial y} \end{vmatrix} \\ & = \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta} \\ \sigma(\boldsymbol{\alpha})=l \\ \sigma(\boldsymbol{\beta})=m}} x^{\sigma_1(\boldsymbol{\alpha}+\boldsymbol{\beta})-\sigma(\boldsymbol{\alpha})+p-1} y^{\sigma_2(\boldsymbol{\alpha}+\boldsymbol{\beta})-\sigma(\boldsymbol{\beta})+q-1} \\ & \qquad \times \begin{vmatrix} M(\boldsymbol{\alpha}) - \widetilde{M}(\boldsymbol{\alpha})\sigma_1(\boldsymbol{\alpha}) & 0 - \widetilde{M}(\boldsymbol{\alpha})\sigma_2(\boldsymbol{\alpha}) \\ 0 - \widetilde{M}(\boldsymbol{\beta})\sigma_1(\boldsymbol{\beta}) & M(\boldsymbol{\beta}) - \widetilde{M}(\boldsymbol{\beta})\sigma_2(\boldsymbol{\beta}) \end{vmatrix} \mathbf{A}^{\boldsymbol{\alpha}}\mathbf{B}^{\boldsymbol{\beta}}, \end{aligned} \tag{2.6}$$

since the conclusion of Theorem 2.6 follows from taking the residue of (2.6) and using Corollary 1.5.

Let $H = 1 + \sum_{i+j\geqslant 1} a_{ij}\, x^i\, y^j$. Then $F = \dfrac{x}{H}$ and so,

$$\frac{\partial F}{\partial x} = \frac{1}{H} - \frac{x}{H^2}\frac{\partial H}{\partial x} = \frac{F}{x} - \frac{F^2}{x}\frac{\partial H}{\partial x}.$$

Hence

$$F^{-l-1}\frac{\partial F}{\partial x} = \frac{F^{-l}}{x} - \frac{F^{-l+1}}{x}\frac{\partial H}{\partial x}. \tag{2.7}$$

By binomial and multinomial expansions, we have

$$\begin{aligned} \left(1 + \sum_{i+j\geqslant 1} a_{ij}x^i y^j\right)^l & = \sum_{k=0}^{l} \binom{l}{k} \left(\sum_{i+j\geqslant 1} a_{ij}x^i y^j\right)^k \\ & = \sum_{k=0}^{l} \binom{l}{k} \sum_{\substack{\boldsymbol{\alpha}' \\ \sigma(\boldsymbol{\alpha}')=k}} \binom{k}{\boldsymbol{\alpha}'} \prod_{i+j\geqslant 1} \left(a_{ij}x^i y^j\right)^{\alpha'_{ij}} \\ & = \sum_{\substack{\boldsymbol{\alpha}' \\ \sigma(\boldsymbol{\alpha}')\leqslant l}} \binom{l}{l-\sigma(\boldsymbol{\alpha}'), \alpha'_{01}, \alpha'_{10}, \ldots, \alpha'_{ij}, \ldots} \prod_{i+j\geqslant 1} \left(a_{ij}x^i y^j\right)^{\alpha'_{ij}} \end{aligned}$$

where $\boldsymbol{\alpha}' = (\alpha'_{ij})_{i+j\geqslant 1}$. Extend $\boldsymbol{\alpha}'$ to $\boldsymbol{\alpha} = (\alpha_{ij})_{i+j\geqslant 0}$ by letting $\alpha_{00} = l - \sigma(\boldsymbol{\alpha}')$ and $\alpha_{ij} = \alpha'_{ij}$ for $i + j \geqslant 1$. Then $\sigma_1(\boldsymbol{\alpha}) = \sigma_1(\boldsymbol{\alpha}')$, $\sigma_2(\boldsymbol{\alpha}) = \sigma_2(\boldsymbol{\alpha}')$ and $\sigma(\boldsymbol{\alpha}) = l$.

Thus the last expression equals

$$\sum_{\substack{\boldsymbol{\alpha} \\ \sigma(\boldsymbol{\alpha})=l}} \binom{l}{\alpha_{00}, \alpha_{01}, \alpha_{10}, \dots, \alpha_{ij}, \dots} \prod_{i+j\geqslant 1} \left(a_{ij} x^i y^j\right)^{\alpha_{ij}}$$

and so

$$F^{-l} = \sum_{\substack{\boldsymbol{\alpha} \\ \sigma(\boldsymbol{\alpha})=l}} M(\boldsymbol{\alpha}) x^{\sigma_1(\boldsymbol{\alpha})-\sigma(\boldsymbol{\alpha})} y^{\sigma_2(\boldsymbol{\alpha})} \mathbf{A}^{\boldsymbol{\alpha}} \tag{2.8}$$

where $\mathbf{A}^{\boldsymbol{\alpha}} = \prod_{i+j\geqslant 1} a_{ij}^{\alpha_{ij}}$.

Now, let $\boldsymbol{\varepsilon}(i,j)$ denote the matrix whose only nonzero entry is 1 and which occurs at the (i,j)-position. Then

$$\frac{\partial H}{\partial x} = \sum_{i+j\geqslant 1} i\, x^{i-1} y^j \mathbf{A}^{\boldsymbol{\varepsilon}(i,j)}.$$

Using (2.8), we see that

$$\frac{F^{-l+1}}{x} \frac{\partial H}{\partial x} = \sum_{i+j\geqslant 1} \sum_{\substack{\boldsymbol{\alpha}' \\ \sigma(\boldsymbol{\alpha}')=l-1}} i\, M(\boldsymbol{\alpha}') x^{\sigma_1(\boldsymbol{\alpha}')-\sigma(\boldsymbol{\alpha}')-1+i-1} y^{\sigma_2(\boldsymbol{\alpha}')+j} \mathbf{A}^{\boldsymbol{\alpha}'} \mathbf{A}^{\boldsymbol{\varepsilon}(i,j)}.$$

For each fixed pair (i,j), let $\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\varepsilon}(i,j)$. Then $\alpha_{ij} \neq 0$, $\sigma_1(\boldsymbol{\alpha}') + i = \sigma_1(\boldsymbol{\alpha})$, $\sigma_2(\boldsymbol{\alpha}') + j = \sigma_2(\boldsymbol{\alpha})$, $\sigma(\boldsymbol{\alpha}') + 1 = \sigma(\boldsymbol{\alpha})$, and $M(\boldsymbol{\alpha}') = \alpha_{ij} \widetilde{M}(\boldsymbol{\alpha})$. Hence

$$\begin{aligned} \frac{F^{-l+1}}{x} \frac{\partial H}{\partial x} &= \sum_{i+j\geqslant 1} \sum_{\substack{\boldsymbol{\alpha} \\ \sigma(\boldsymbol{\alpha})=l \\ \alpha_{ij}\neq 0}} i\, \alpha_{ij}\, \widetilde{M}(\boldsymbol{\alpha})\, x^{\sigma_1(\boldsymbol{\alpha})-\sigma(\boldsymbol{\alpha})-1}\, y^{\sigma_2(\boldsymbol{\alpha})} \mathbf{A}^{\boldsymbol{\alpha}} \\ &= \sum_{\substack{\boldsymbol{\alpha} \\ \sigma(\boldsymbol{\alpha})=l}} \widetilde{M}(\boldsymbol{\alpha})\, \sigma_1(\boldsymbol{\alpha})\, x^{\sigma_1(\boldsymbol{\alpha})-\sigma(\boldsymbol{\alpha})-1}\, y^{\sigma_2(\boldsymbol{\alpha})} \mathbf{A}^{\boldsymbol{\alpha}}. \end{aligned}$$

The last expression is obtained by noting the condition $\alpha_{ij} \neq 0$ for the inner sum is redundant as α_{ij} is a factor of the summand, and using the definition of $\sigma_1(\boldsymbol{\alpha})$ after the summation signs are interchanged. Hence it follows from (2.7) and (2.8) that

$$F^{-l-1} \frac{\partial F}{\partial x} = \sum_{\substack{\boldsymbol{\alpha} \\ \sigma(\boldsymbol{\alpha})=l}} \Big[M(\boldsymbol{\alpha}) - \widetilde{M}(\boldsymbol{\alpha}) \sigma_1(\alpha) \Big] x^{\sigma_1(\boldsymbol{\alpha})-\sigma(\boldsymbol{\alpha})-1} y^{\sigma_2(\boldsymbol{\alpha})} \mathbf{A}^{\boldsymbol{\alpha}}. \tag{2.9}$$

Since

$$\frac{\partial F}{\partial y} = 0 - \frac{F^2}{x} \frac{\partial H}{\partial y},$$

we may proceed as before to obtain

$$F^{-l-1} \frac{\partial F}{\partial y} = \sum_{\substack{\boldsymbol{\alpha} \\ \sigma(\boldsymbol{\alpha})=l}} \Big[0 - \widetilde{M}(\boldsymbol{\alpha}) \sigma_2(\alpha) \Big] x^{\sigma_1(\boldsymbol{\alpha})-\sigma(\boldsymbol{\alpha})} y^{\sigma_2(\boldsymbol{\alpha})-1} \mathbf{A}^{\boldsymbol{\alpha}}. \tag{2.10}$$

Likewise, we have

$$G^{-m} = \sum_{\substack{\boldsymbol\beta \\ \sigma(\boldsymbol\beta)=m}} M(\boldsymbol\beta)\, x^{\sigma_1(\boldsymbol\beta)}\, y^{\sigma_2(\boldsymbol\beta)-\sigma(\boldsymbol\beta)} \mathbf{B}^{\boldsymbol\beta},$$

$$G^{-m-1}\frac{\partial G}{\partial x} = \sum_{\substack{\boldsymbol\beta \\ \sigma(\boldsymbol\beta)=m}} \Big[0 - \widetilde{M}(\boldsymbol\beta)\sigma_1(\beta)\Big] x^{\sigma_1(\boldsymbol\beta)-1} y^{\sigma_2(\boldsymbol\beta)-\sigma(\boldsymbol\beta)} \mathbf{B}^{\boldsymbol\beta}, \tag{2.11}$$

$$G^{-m-1}\frac{\partial G}{\partial y} = \sum_{\substack{\boldsymbol\beta \\ \sigma(\boldsymbol\beta)=m}} \Big[M(\boldsymbol\beta) - \widetilde{M}(\boldsymbol\beta)\sigma_2(\beta)\Big] x^{\sigma_1(\boldsymbol\beta)} y^{\sigma_2(\boldsymbol\beta)-\sigma(\boldsymbol\beta)-1} \mathbf{B}^{\boldsymbol\beta}. \tag{2.12}$$

Now (2.6) follows from (2.9), (2.10), (2.11) and (2.12). □

Corollary 2.7. *For any* $(\boldsymbol\alpha, \boldsymbol\beta) \in \mathcal{F}_{p,q}$,

$$R(\boldsymbol\alpha,\boldsymbol\beta) = \left| \begin{pmatrix} M(\boldsymbol\alpha) & 0 \\ 0 & M(\boldsymbol\beta) \end{pmatrix} - \begin{pmatrix} \widetilde{M}(\boldsymbol\alpha) & 0 \\ 0 & \widetilde{M}(\boldsymbol\beta) \end{pmatrix} \begin{pmatrix} \sigma_1(\boldsymbol\alpha) & \sigma_2(\boldsymbol\alpha) \\ \sigma_1(\boldsymbol\beta) & \sigma_2(\boldsymbol\beta) \end{pmatrix} \right|.$$

Proof. Compare the results of Theorems 2.5 and 2.6. □

Remark. Using Corollary 2.7 we can now determine the $e^{p,q}_{l,m}$ in Theorem 2.4.

3. Generalizations

Using essentially the same proofs, all results in Section 2 can be generalized to an arbitrary number of colors. In what follows, we shall indicate how this can be done.

An *n-colored tree* is a (finite) rooted tree in which every node has one of the n colors and the children of each node are linearly ordered in such a way that children of color i always precede those of color j if $i < j$. A *$(p_1, \ldots, p_n)$-forest* is an ordered set of n-colored trees with p_i trees of root-color i for $i = 1, \ldots, n$ such that trees with root-color i always precede those of root-color j if $i < j$.

For $i = 1, \ldots, n$, let $\boldsymbol\alpha^{(i)} = (\alpha^{(i)}_{k_1,\ldots,k_n})_{k_1,\ldots,k_n \in \mathbb{N}}$ be an n-dimensional array over $\mathbb{N}$ with finitely many nonzero entries. If, for each i and for each n-tuple $(k_1, \ldots, k_n)$, $\alpha^{(i)}_{k_1,\ldots,k_n}$ equals the number of color i nodes in a $(p_1, \ldots, p_n)$-forest F with k_j children of color j for each $j = 1, \ldots, n$, then $\widehat{\boldsymbol\alpha} = (\boldsymbol\alpha^{(1)}, \ldots, \boldsymbol\alpha^{(n)})$ is said to be the *inventory* of F.

For any n-dimensional array $\boldsymbol\alpha = (\alpha_{k_1,\ldots,k_n})_{k_1,\ldots,k_n \in \mathbb{N}}$ over $\mathbb{N}$ with finitely many nonzero entries, define $\sigma(\boldsymbol\alpha) = \sum \alpha_{k_1,\ldots,k_n}$ and $\sigma_i(\boldsymbol\alpha) = \sum k_i \alpha_{k_1,\ldots,k_n}$, for each i.

As in the case of Lemma 2.1, if $\widehat{\boldsymbol\alpha} = (\boldsymbol\alpha^{(1)}, \ldots, \boldsymbol\alpha^{(n)})$ is the inventory of a $(p_1, \ldots, p_n)$-forest, then, for each i,

$$p_i = \sigma(\boldsymbol\alpha^{(i)}) - \sigma_i(\boldsymbol\alpha^{(1)} + \cdots + \boldsymbol\alpha^{(n)}). \tag{3.1}$$

Given an n-tuple $\widehat{\boldsymbol\alpha} = (\boldsymbol\alpha^{(1)}, \ldots, \boldsymbol\alpha^{(n)})$ of n-dimensional arrays, let $R(\widehat{\boldsymbol\alpha})$ be the number of $(p_1, \ldots, p_n)$-forests with inventory $\widehat{\boldsymbol\alpha}$, where each p_i is defined by (3.1). For $p_1, \ldots, p_n \in \mathbb{N}$, we shall also denote by $\mathcal{F}_{p_1,\ldots,p_n}$ the set of all $\widehat{\boldsymbol\alpha} = (\boldsymbol\alpha^{(1)}, \ldots, \boldsymbol\alpha^{(n)})$ satisfying (3.1). For each i, let X_i be the generating function for the number of

$\mathbf{e}_i$-forests classified by inventories. Here, $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the 1 being in the i^{th} position. Then

$$X_i = \sum_{\mathcal{F}_{\mathbf{e}_i}} R(\widehat{\boldsymbol{\alpha}}) \prod_{j=1}^{n} \prod_{k_1,k_2,\ldots,k_n} {a^{(j)}_{k_1,k_2,\ldots,k_n}}^{\alpha^{(j)}_{k_1,k_2,\ldots,k_n}}.$$

One can show, as in Lemma 2.2, that $X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}$ is the generating function for the number of $(p_1, \ldots, p_n)$-forests classified by inventories. Thus, as in Lemma 2.3, X_i satisfies the functional equation

$$X_i = \sum a^{(i)}_{k_1,\ldots,k_n} X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n},$$

for $i = 1, \ldots, n$.

We can now state our main results for the general case of arbitrary n.

Theorem 3.1 (Direct Reversion). *For each $i = 1, \ldots, n$, let*

$$F_i = x_i - \sum_{k_1+\cdots+k_n \geqslant 2} a^{(i)}_{k_1,\ldots,k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},$$

where the $a^{(i)}_{k_1,\ldots,k_n}$ are indeterminates. Then, for $p_1, \ldots, p_n \in \mathbb{N}$,

$$x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} = \sum_{r_1,\ldots,r_n} e^{p_1,\ldots,p_n}_{r_1,\ldots,r_n} F_1^{r_1} F_2^{r_2} \cdots F_n^{r_n},$$

where

$$e^{p_1,\ldots,p_n}_{r_1,\ldots,r_n} = \sum R(\widehat{\boldsymbol{\alpha}}) \prod_{j=1}^{n} \prod_{k_1+\cdots+k_n \geqslant 2} {a^{(j)}_{k_1,\ldots,k_n}}^{\alpha^{(j)}_{k_1,\ldots,k_n}},$$

with the sum indexed by all $\widehat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)})$ in $\mathcal{F}_{p_1,\ldots,p_n}$ satisfying the conditions that $r_i = \alpha^{(i)}_{0,\ldots,0}$ and $\alpha^{(i)}_{\mathbf{e}_j} = 0$ for all i and j.

Theorem 3.2 (Diagonal Reversion). *For each $i = 1, \ldots, n$, let*

$$F_i = \frac{x_i}{1 + \sum\limits_{k_1+\cdots+k_n \geqslant 1} a^{(i)}_{k_1,\ldots,k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}.$$

Then, for $p_1, \ldots, p_n \in \mathbb{N}$,

$$x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} = \sum_{r_1,\ldots,r_n} d^{p_1,\ldots,p_n}_{r_1,\ldots,r_n} F_1^{r_1} F_2^{r_2} \cdots F_n^{r_n},$$

where

$$d^{p_1,\ldots,p_n}_{r_1,\ldots,r_n} = \sum R(\widehat{\boldsymbol{\alpha}}) \prod_{j=1}^{n} \prod_{k_1+\cdots+k_n \geqslant 1} {a^{(j)}_{k_1,\ldots,k_n}}^{\alpha^{(j)}_{k_1,\ldots,k_n}},$$

with the sum indexed by all $\widehat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)})$ in $\mathcal{F}_{p_1,\ldots,p_n}$ satisfying the conditions that $r_i = \sigma(\boldsymbol{\alpha}^{(i)})$, for all i.

Theorem 3.3 (Extended Raney Coefficient). *For any* $\widehat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}) \in \mathcal{F}_{p_1,\ldots,p_n}$,

$$R(\widehat{\boldsymbol{\alpha}}) = \left| \begin{pmatrix} M(\boldsymbol{\alpha}^{(1)}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & M(\boldsymbol{\alpha}^{(n)}) \end{pmatrix} - \begin{pmatrix} \widetilde{M}(\boldsymbol{\alpha}^{(1)}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{M}(\boldsymbol{\alpha}^{(n)}) \end{pmatrix} \begin{pmatrix} \sigma_1(\boldsymbol{\alpha}^{(1)}) & \ldots & \sigma_n(\boldsymbol{\alpha}^{(1)}) \\ \vdots & \ddots & \vdots \\ \sigma_1(\boldsymbol{\alpha}^{(n)}) & \ldots & \sigma_n(\boldsymbol{\alpha}^{(n)}) \end{pmatrix} \right|.$$

Corollary 3.4. *Let* $\widehat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}) \in \mathcal{F}_{p_1,\ldots,p_n}$. *If all* $p_i > 0$, *then* $R(\widehat{\boldsymbol{\alpha}}) > 0$.

Proof. If each $p_i > 0$, then, by (3.1), none of $\boldsymbol{\alpha}^{(1)}, \ldots, \boldsymbol{\alpha}^{(n)}$ consists entirely of 0's; hence, $M(\boldsymbol{\alpha}^{(i)}) = \sigma(\boldsymbol{\alpha}^{(i)})\widetilde{M}(\boldsymbol{\alpha}^{(i)})$, for all i. Thus, by Theorem 3.3,

$$R(\widehat{\boldsymbol{\alpha}}) = \prod_{i=1,\ldots,n} \widetilde{M}(\boldsymbol{\alpha}^{(i)}) \times \begin{vmatrix} \sigma(\boldsymbol{\alpha}^{(1)}) - \sigma_1(\boldsymbol{\alpha}^{(1)}) & -\sigma_2(\boldsymbol{\alpha}^{(1)}) & \ldots & -\sigma_n(\boldsymbol{\alpha}^{(1)}) \\ -\sigma_1(\boldsymbol{\alpha}^{(2)}) & \sigma(\boldsymbol{\alpha}^{(2)}) - \sigma_2(\boldsymbol{\alpha}^{(2)}) & \ldots & -\sigma_n(\boldsymbol{\alpha}^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_1(\boldsymbol{\alpha}^{(n)}) & -\sigma_2(\boldsymbol{\alpha}^{(n)}) & \ldots & \sigma(\boldsymbol{\alpha}^{(n)}) - \sigma_n(\boldsymbol{\alpha}^{(n)}) \end{vmatrix}.$$

The result now follows from Lemma 3.5 below. □

Lemma 3.5. *Suppose* B *is a square matrix of order* n *with real entries* b_{ij} *such that* $b_{ii} \geqslant 0$, *for all* i, *and* $b_{ij} \leqslant 0$, *for* $i \neq j$. *If all the column-sums of* B *are nonnegative, then so is the determinant of* B. *If, furthermore, all the column-sums of* B *are positive, then so is the determinant of* B.

Proof. By hypothesis, the j^{th} column-sum $p_j = \sum_{i=1}^n b_{ij} \geqslant 0$, for all j. Thus we can write B as a sum of two matrices with one of them diagonal, namely,

$$\begin{aligned} B &= P + B' \\ &= \begin{pmatrix} p_1 & & & \mathbf{0} \\ & p_2 & & \\ & & \ddots & \\ \mathbf{0} & & & p_n \end{pmatrix} + \begin{pmatrix} -\sum\limits_{k\neq 1} b_{k1} & b_{12} & \ldots & b_{1n} \\ b_{21} & -\sum\limits_{k\neq 2} b_{k2} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & -\sum\limits_{k\neq n} b_{kn} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \det B &= p_1 p_2 \cdots p_n + \left(\sum_{1\leqslant i_1 < i_2 < \cdots < i_l \leqslant n-1} c_{i_1,\ldots,i_l} p_{i_1} p_{i_2} \cdots p_{i_l} \right) + \det B' \\ &= p_1 p_2 \cdots p_n + \sum_{1\leqslant i_1 < i_2 < \cdots < i_l \leqslant n-1} c_{i_1,\ldots,i_l} p_{i_1} p_{i_2} \cdots p_{i_l}, \end{aligned}$$

a polynomial in $p_1, \ldots, p_n$, where $c_{i_1,\ldots,i_l}$ is the principal minor of B' obtained by deleting the i_j^{th} row and column of B', for $j = 1, 2, \ldots, l$. We only need to show that $c_{i_1,\ldots,i_l} \geqslant 0$, for all $l = 1, 2, \ldots, n-1$. However, this follows from the induction hypothesis, since B' as well as all its principal submatrices satisfy the hypotheses of Lemma 3.5. □

References

1. H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: Reduction of degree and formal expansion of the inverse*, Bulletin (New Series) of the American Mathematical Society **7** (1982), 287–330, MR **83k:**14028. Zbl.539.13012.
2. N. Bourbaki, *Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1972, MR **50:**12997. Zbl.279.13001.
3. I. J. Good, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes*, Proceedings of the Cambridge Philosophical Society **56** (1960), 367–380, MR **23:**A352. Zbl.135.18802.
4. M. Haiman and W. Schmitt, *Incidence algebra antipodes and Lagrange inversion in one and several variables*, Journal of Combinatorial Theory, Series A **50** (1989), 172–185, MR **90f:**05005. Zbl.747.05007.
5. C. G. J. Jacobi, *De resolutione aequationum per series infinitas*, Journal für die reine und angewandte Mathematik **6** (1830), 257–286.
6. D. E. Knuth, *The Art of Computer Programming, Vol. 1:* Fundamental Algorithms, second edition, Addison-Wesley, Reading, Massachusetts, 1973, MR **44:**3530; MR **51:**14624; MR **57:**18169.
7. G. N. Raney, *Functional composition patterns and power series reversion*, Transactions of the American Mathematical Society **94** (1960), 441–451, MR **22:**5584. Zbl.131,14.
8. J. Towber, *A combinatorial conjecture which implies the Jacobian conjecture* (to appear).
9. S. S.-S. Wang, *A Jacobian criterion for separability*, Journal of Algebra **65** (1980), 453–494, MR **83e:**14010. Zbl.471.13005.
10. D. Wright, *The tree formulas for reversion of power series*, Journal of Pure and Applied Algebra **57** (1989), 191–211, MR **90d:**13008. Zbl.672.13010.

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